Heat-Transfer Analysis of the Basal Melting of Antarctic Ice Shelves

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Basal melting of Antarctic ice shelves is an important element in the overall balance of Antarctic ice. A heat-transfer model for the basal melting of the Drygalski Ice Tongue is presented. The model does not contain any adjustable parameter. The calculated basal melting rate agrees very well with the value estimated from an overall ice balance on the ice tongue. It is concluded that relatively simple concepts of transport phenomena may be used to model some important features of the dynamics of the Antarctic ice sheet.

Introduction

It is estimated that 90% of the sweet water of the Earth is in the Antarctic; should all this ice melt, the average sea level would increase by about 60 m (Robin, 1979). This shows clearly that, to avoid major disruptions of the planetary climatology, the delicate balance between ice accumulation and depletion in the Antarctic is to be kept quite precisely.

This is an important problem in the global planetary ecology, and environmental problems are certainly of interest to chemical engineers. Use of classical concepts of chemical engineering science can usefully be brought to bear on the analysis of at least some features of the global ice balance of the Antarctic.

Antarctic glaciers discharge ice into floating (or outlet) glaciers called ice shelves, which occupy about 10% of the surface area of the ice sheet (Thomas, 1979). Ice shelves exist also in Greenland, where in fact they were first recognized (Rink, 1877); the first major contribution to the understanding of Antarctic ice shelves is due to Wright and Priestley (1922) who reported results of Scott's last expedition. Some ice shelves take the form of ice tongues, roughly rectangular formations bounded by water on three sides. The Drygalski Ice Tongue, shown in a satellite photograph in Figure 1, is the "floating" part of the David Glacier, the most important glacier of the Northern Victoria territory in the Antarctic. This glacier drains an area of about 224,000 km^2. A large amount of data on the Drygalski Ice Tongue is available; a very concise review of this information is given in the following.

The Drygalski Ice Tongue is, at the time of writing, approximately 70 km long and 20 km wide. The thickness of the ice layer has been measured by radio-echo sounding by the Scott Polar Research Institute; at the beginning of the DIT (the glacier side), it is approximately 1 km. Some data on the thickness at different positions along the tongue are reported in Figure 2; these show that the vertical cross section is approximately triangular. Surface velocities of the tongue have been measured (Landsat 1 MSS 1973–Landsat 4 TM 1988). Since the Drygalski Ice Tongue rests on an essentially frictionless bed of water, surface velocities are presumably good representations of the velocity at all depths (Thomas, 1979). At the glacier side, the velocity is 700 m/yr, and it slowly increases toward a value of 900 m/yr at the tip. The values at the glacier side allow an estimate of the ice flow rate into the tongue of 14 km^3/yr. To get a feeling for the magnitude of this flow rate, the Amazon river, which is the largest river in the world, has a flow rate which is 300 times as large. The velocity in the Amazon river is 30,000 times as large as that in the Drygalski ice tongue, but the cross-sectional area is 100 times smaller.

The area of the Drygalski Ice Tongue has increased significantly over the last 30 years, see Figure 2: the tip has advanced at about 1 km/yr between 1960 and 1990; no major icebergs have detached from it during the same period. If one assumes that the vertical cross section of the Drygalski Ice Tongue is approximately triangular, this corresponds to a increase rate of the total volume of 10 km^3/yr. If the two estimates given above are correct, one concludes that at the present time the net rate of loss of ice from the Drygalski Ice Tongue is 4 km^3/yr.

It is also of interest to note that the problem is clearly an unsteady-state one and that a globally cyclic behavior can be achieved only by occasional shedding of major icebergs. None of these have been reported over the last 30 years, indicating that the time over which reliable observational data are avail-

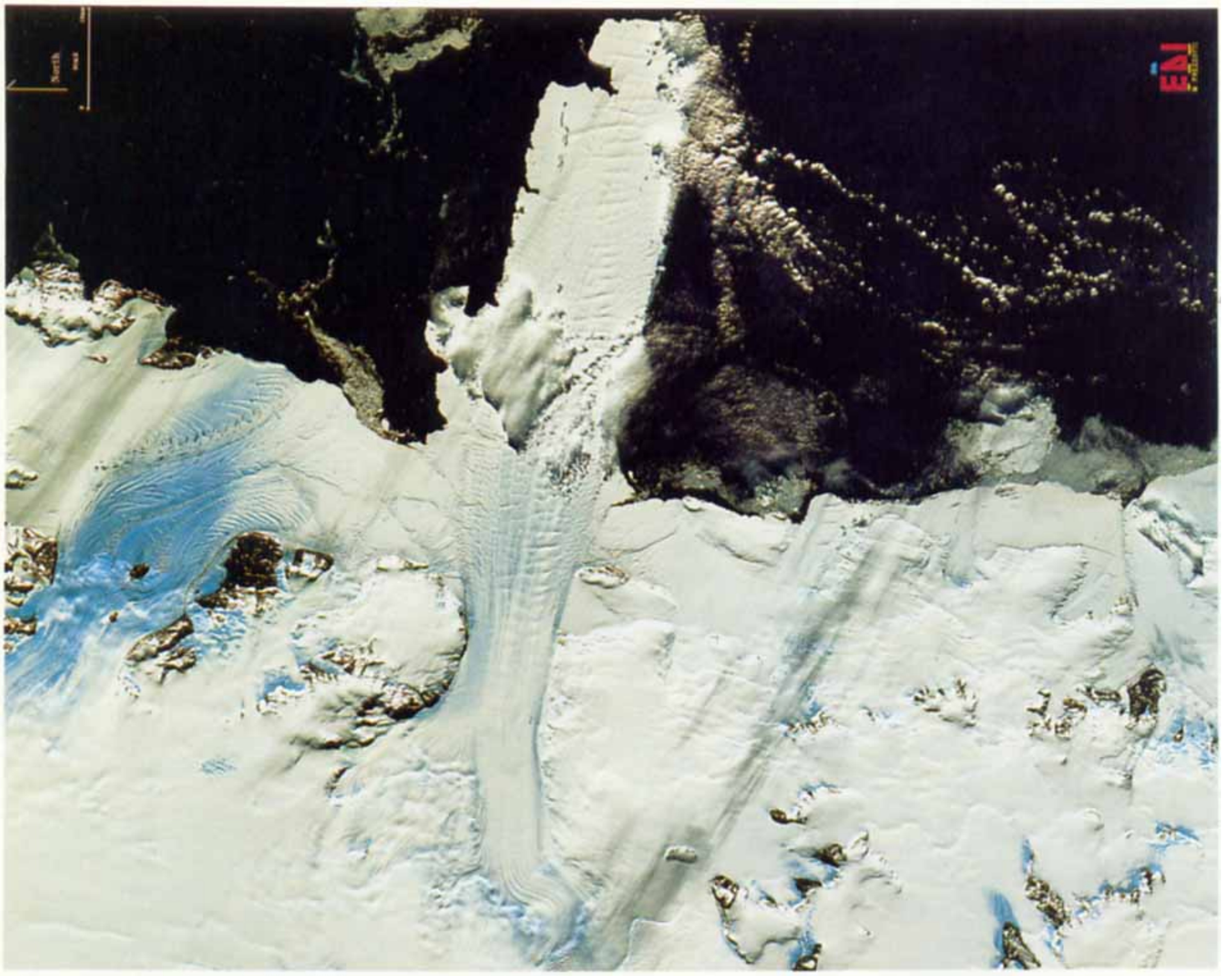

Figure 1. Satellite photograph of the Drygalski Ice Tongue.

able is only a fraction of a total cycle time. The global unsteadiness of the problem is discussed in more detail in the Appendix.

The net rate of loss is the difference between the rate of basal melting (melting from the lower surface of the tongue which is exposed to seawater) and the net rate of accumulation on its exposed surface. The latter is due to three effects: ablation due to catabatic winds, sublimation and snowfall. These three effects are hard to estimate separately, but the net rate of gain of ice on the surface can be estimated as follows. Since the tongue drains an area of 224,000 km^2, the 14 km^3/yr represents an average deposition rate of 0.62×10^{-4} km/yr over the drainage area. If one assumes this rate to apply also on the upper surface of the Drygalski Ice Tongue, which has an area of 1,400 km^2, one estimates a total deposition rate of 0.087 km^3/yr. This is only 2% of the net rate of loss, and therefore basal melting essentially accounts for the entire loss rate of ice.

While data on ice thickness, velocity, flow rate, and so on are abundant, there seems to be no model in the literature which quantifies the melting rate of the Drygalski Ice Tongue or, for that matter, of any Antarctic ice shelf. This article presents a crude preliminary heat-transfer model for the basal melting of the Drygalski Ice Tongue, which contains no adjustable parameters. As will be seen, the model predicts the correct order of magnitude of the relevant macroscopically observable variables.

Conditions at the Glacier Side

Consider the ice layer at the glacier side of the Drygalski Ice Tongue, see Figure 3. The ice comes from the David Glacier, and it has been exposed to the annual cycles of temperature for a very long time. Consider a thick slab of solid, with heat diffusivity α, which on the exposed surface is exposed to temperature cycles of frequency Ω. The amplitude of the temperature cycles decreases with distance from the exposed surface (Carlslaw and Jaeger, 1959); the cycles are damped to a negligible fraction of their surface amplitude at a depth of the order of $\sqrt{\alpha/\Omega}$. The heat diffusivity α in ice is 1.25×10^{-6} m^2/s; it follows that, with $\Omega = 1$ yr^{-1}, damping occurs over a few tens of meters: to all practical purposes, the ice layer at the beginning of the Drygalski Ice Tongue may be regarded as being uniformly at the average yearly temperature of the Antarctic, which is only of the order of $-35°C$.

The phenomenon near the glacier side of the DIT can thus be modeled as a semiinfinite slab of ice, initially at the uniform temperature $T_0 = -35°C$. At time $t = 0$, surface $y^* = 0$ is sud-

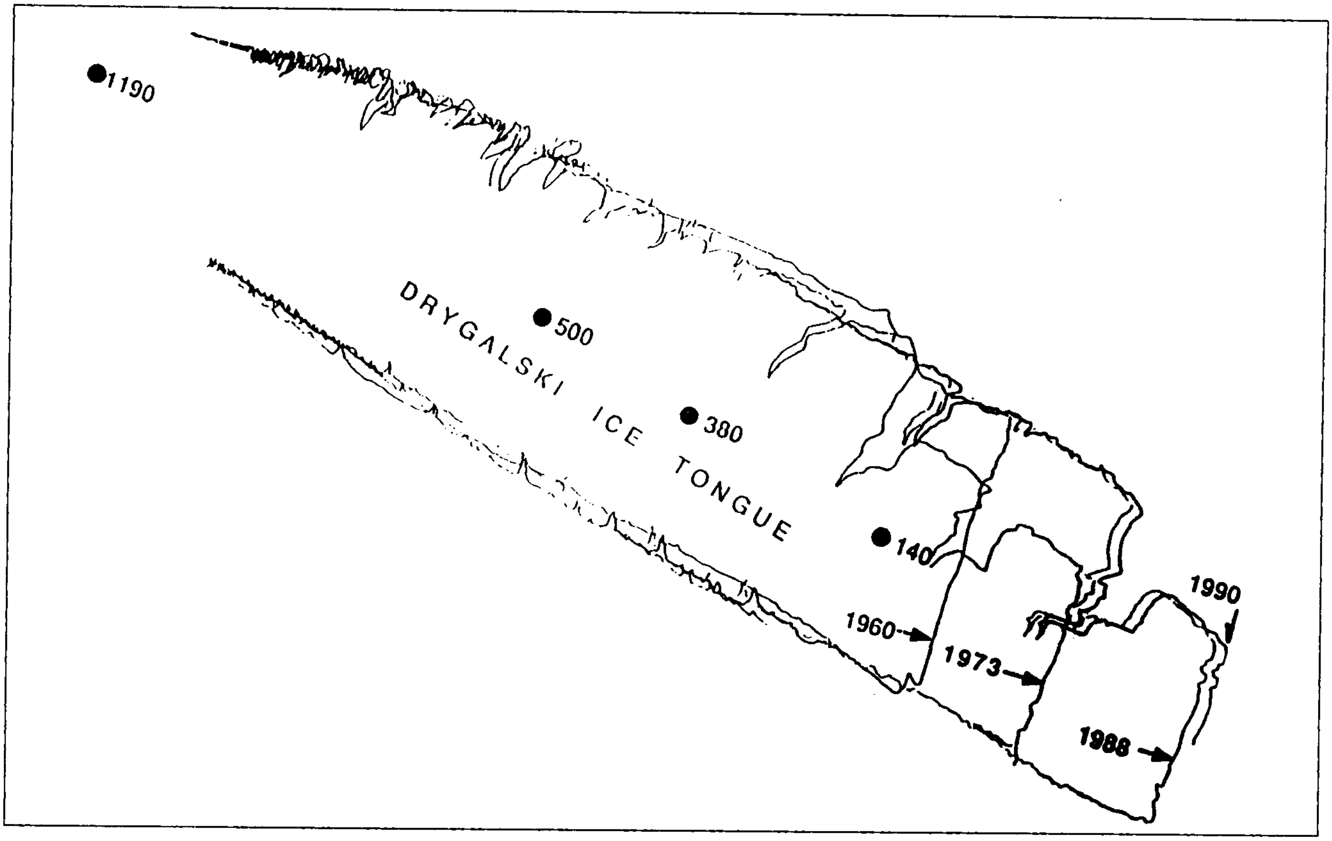

Figure 2. Surface of the Drygalski Ice Tongue as determined by aerophotogrammetry in the last 30 years.

denly exposed to water. We begin by considering the case where the seawater is at exactly the equilibrium freezing temperature T^*. Freezing will occur at the exposed surface, the liberated latent heat being carried by unsteady heat conduction into the ice layer. If the y^* coordinate points out of the ice layer, the interface will be located at $y^*=s(t)$, $s(0)=0$, where s is the thickness of the additional ice layer resulting from the freezing phenomenon up to time t.

The problem as formulated above is a free boundary problem (Fasano and Primicerio, 1983), which turns out to be a minor variation of the classical Stefan (1891) problem. With $\vartheta=(T-T_0)/(T^*-T_0)$ the dimensionless (internal) temperature, Γ the latent heat, and c the specific heat of ice, the differential equations and boundary conditions of the problem are:

$$\alpha\frac{\partial^2\vartheta}{\partial y^{*2}}=\frac{\partial\vartheta}{\partial t} \tag{1}$$

$$t=0,\quad \vartheta=0 \tag{2}$$

$$y^*=s(t),\quad \vartheta=1 \tag{3}$$

$$y^*=s(t),\quad \alpha\frac{\partial\vartheta}{\partial y^*}=St\frac{ds}{dt} \tag{4}$$

$$s(0)=0 \tag{5}$$

The dimensionless group St is the Stefan number, $St=\Gamma/c(T^*-T_0)$. (In the Russian literature, $1/St$ is often called the Kutatedadze number.) The problem, as formulated, does not possess an external length scale, and hence the solution is necessarily expressible in terms of ϑ being a unique function of the Neuman variable $y^*/\sqrt{\alpha t}$. The solution is:

$$s(t)=2K\sqrt{\alpha t} \tag{6}$$

$$\vartheta=\frac{1+\operatorname{erf}\left(\dfrac{y^*}{2\sqrt{\alpha t}}\right)}{1+\operatorname{erf}K} \tag{7}$$

where K is implicitly given by:

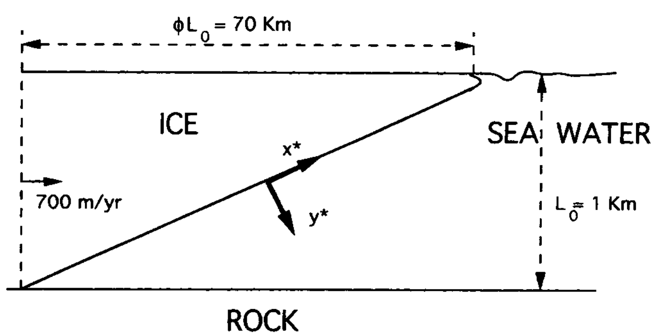

Figure 3. Cross section of the Drygalski Ice Tongue.
Coordinates used in the modeling are indicated.

$$K \exp(K^2)(1+\text{erf } K)=\frac{1}{St\sqrt{\pi}} \tag{8}$$

As an aside, it is worthwhile to point out some interesting features of the problem discussed above. The dimensionless temperature at $y^*=0$, that is, at the original position of the surface, is constant in time at a value $1/(1+\text{erf } K)$. At the same position, there is an inflection point of the $\vartheta(y^*)$ curve. Since ϑ is constant at $y^*=0$, the solution at $y^*>0$ is exactly the same one would obtain for the classical Stefan problem, should one use in the latter a driving force equal to erf $K/(1+\text{erf } K)$ of the total one; the solution at $y^*<0$ is the classical penetration theory (Higbie, 1935) solution for a dimensionless driving force $1/(1+\text{erf } K)$.

To calculate the value of the Stefan number, one needs to know the value of T^*. This will be discussed in more detail later; here it is enough to note that T^* does not differ from 0°C by more than at most 2°C, and hence that, for an initial estimate, one may take T^*-T_0 as 35°C. With this, $St=4.56$, and the corresponding value of the constant K is 0.109.

The seawater temperature, however, is somewhat larger than T^*, and hence the problem needs to be reformulated, as discussed in the next section; as will be seen, Eqs. 6 and 7 are the asymptotic solution for small distances from the glacier side of the Drygalski Ice Tongue.

Competition Between Freezing and Melting

Let h be the heat-transfer coefficient in the seawater at the ice surface. With T_W different from T^*, heat conduction into the ice layer must remove not only the liberated latent heat, but also the heat convected to the surface by the warmer seawater. Hence, boundary condition 4 becomes:

$$y^*=s(t), \quad \alpha\frac{\partial\vartheta}{\partial y^*}=H\epsilon+St\frac{ds}{dt} \tag{9}$$

where H is the heat-transfer coefficient h divided by the product of ice density and specific heat, $H=h/\rho c$, and ϵ is the ratio of the external and internal temperature driving forces:

$$\epsilon=\frac{T_W-T^*}{T^*-T_0} \tag{10}$$

The appearance of the term $H\epsilon$ in Eq. 9 now provides an external length scale α/H (note that, with k the thermal conductivity of ice, $\alpha/H=k/h$). Hence, a similarity solution in terms of the Neuman variable is not available for the complete problem. It is also of interest that, given the absence of a geometrical length scale, no Biot number in the classical form can be defined. The only length scale with which k/h can be compared is the internal one $\sqrt{\alpha t}$, so that one obtains a time-dependent Biot number $h\sqrt{\alpha t}/k$. This guarantees that at short times the external resistance to heat transfer is predominant over the internal one, that is, Eq. 4 is appropriate in a neighborhood of time zero.

The problem formulation can be made dimensionless by using k/h as the length scale and $k^2/\alpha h^2$ as the time scale. The following dimensionless variables are thus introduced:

$$u=\frac{y^*h}{k} \tag{11}$$

$$\tau=\frac{\alpha th^2}{k^2} \tag{12}$$

$$\mu(\tau)=\frac{s(t)h}{k} \tag{13}$$

The dimensionless formulation of the problem thus becomes:

$$\frac{\partial^2\vartheta}{\partial u^2}=\frac{\partial\vartheta}{\partial\tau} \tag{14}$$

$$\tau=0, \quad \vartheta=0 \tag{15}$$

$$u=\mu(\tau), \quad \vartheta=1 \tag{16}$$

$$u=\mu(\tau), \quad \frac{\partial\vartheta}{\partial u}=\epsilon+St\frac{d\mu}{d\tau} \tag{17}$$

$$\tau=0, \quad \mu=0 \tag{18}$$

While the actual value of T_W-T^* has not been discussed yet, its order of magnitude is 0.5°C, and hence ϵ is a small parameter ($\epsilon\approx0.01$). It thus seems natural to seek a solution by means of a perturbation expansion, with ϵ itself being the perturbation parameter. However, classical perturbation procedures (Nayfeh, 1973) cannot usefully be applied because the perturbation parameter appears in the nastiest place, Eq. 17, which is at the same time a boundary condition for Eq. 14 and a differential equation for $\mu(\tau)$.

The perturbation approach is not, however, totally useless, since one immediately realizes that the solution of the subproblem discussed in the previous section is the order-zero solution of the complete problem. To within order zero in ϵ one has:

$$u=\mu(\tau), \quad \frac{\partial\vartheta}{\partial u}=\frac{KSt}{\sqrt{\tau}} \tag{19}$$

It follows that, in a neighborhood of time zero, the lefthand side of Eq. 17 is guaranteed to be much larger than ϵ, and hence that the order zero solution indeed holds at short times. Equation 19 also shows that the lefthand side of Eq. 17 decays in time, and hence that at some critical time τ_C, it will become less than ϵ. At τ values exceeding τ_C, ds/dt becomes negative, and hence melting of ice will begin to occur. The value of τ_C can be estimated by requiring the lefthand side of Eq. 17, as estimated from Eq. 19, to become equal to ϵ:

$$\tau_C=\left(\frac{KSt}{\epsilon}\right)^2=1{,}210 \tag{20}$$

A strictly analogous problem was attacked by Epstein (1976) using an integral method. The value of τ_C can be read off the results of Epstein as the time at which $\mu(\tau)$ reaches a maximum; the value obtained is very close to 1,210, the difference being within the uncertainty related to the necessary interpolations between the reported curves.

To estimate the dimensional critical time $t_C = k^2\tau_C/h^2\alpha$, we need to estimate the heat-transfer coefficient h in the seawater. This is discussed in the next section.

Heat-Transfer Coefficient

Heat transfer in the water may be due to forced convection (related to the average seawater velocity due to both prevailing currents and to tidal pumping), as well as to free convection. The latter is induced by two effects: density differences due to the temperature difference $T_W - T^*$ and density differences due to differences in salinity (when melting occurs, the water in immediate contact with the ice layer is sweet water). The latter effect is by far the predominant one: salt water has a density of 1,023 kg/m^3 so that the density difference is of the order of 20 kg/m^3. This is significantly more than the density difference due to a 0.5°C temperature difference in the region close to the maximum in the density vs. temperature plot for water.

Because the water in immediate contact with the ice surface is almost fresh water, at a salinity corresponding to the local freezing temperature, a salt diffusion phenomenon should be taken into account in this analysis. A study we have almost completed gives us confidence to say that this phenomenon can be neglected. In addition, in this article we are interested only in an order of magnitude analysis, and therefore the salt diffusion can safely be neglected.

The fact that free convection is driven by a salinity-induced density difference is fortunate. In fact, since water has its maximum density at 3.98°C, free convection driven by temperature-induced density differences presents a number of problems (Bendell and Gebhart, 1976). No such difficulties arise in the problem at hand, where the water temperatures are invariably below the point of maximum density. Furthermore, since the density difference is due to salinity, the problem at hand is a rare example of a *linear* free convection problem, where the heat flux is proportional to the thermal driving force.

In their comprehensive review of solidification and melting, Cheung and Epstein (1982) concluded that "a refined (free) convection model is needed to describe more realistically the unstable situation of melting from below. A survey of the literature failed to unearth such a model." In our case, however, the melting surface is not exactly horizontal, and the free convection phenomenon can be attacked via a scaling argument which can be traced back to the work of Hermann (1936); the argument was developed in detail by Acrivos (1960) and Acrivos et al. (1960), and was later extended to different contexts by Ng and Hartnett (1988) and by Ocone and Astarita (1991). With reference to Figure 3, let x^* be a coordinate along the ice surface, and y^* the coordinate orthogonal to it; let v^* and w^* be the x^* and y^* components of velocity. Let L_0 be the vertical depth of the water layer (≈ 1 km), and ΦL_0 the length along the x^* direction (≈ 70 km). ΦL_0 is obviously the appropriate scale in the x^* direction, and hence we define $x = x^*/\Phi L_0$ so that x is of order unity. Note that (see the Appendix), the inclination Φ is in fact a function of x; here we use its average value for an order of magnitude analysis.

Having established the appropriate scale for distances along the x^* direction, we also need appropriate scales for v^*, w^*, and y^*, say V, W, and Y such that $v = v^*/V$, $w = w^*/W$, and $y = y^*/Y$ are of order unity. Note that Y is the thickness of the thermal boundary layer, and therefore an estimate of Y is tantamount to an estimate of the heat-transfer coefficient.

The continuity equation establishes a first relationship between the scales, say:

$$\frac{V}{\Phi L_0} = \frac{W}{L_0} \tag{21}$$

so that the dimensionless continuity equation takes the following parameter-free form:

$$\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} = 0 \tag{22}$$

The energy balance equation now becomes, with $\beta = (T - T^*)/(T_W - T^*)$, the (external) dimensionless temperature and α_W the heat diffusivity in water:

$$v\frac{\partial \beta}{\partial x} + w\frac{\partial \beta}{\partial y} = \frac{\alpha_W}{WY}\frac{\partial^2 \beta}{\partial y^2} \tag{23}$$

Since near the ice surface conduction and convection are necessarily of the same order of magnitude, a second relationship between the scales is established:

$$WY = \alpha_W \tag{24}$$

Now let $\Delta\rho$ be the density difference which drives the free convection flow, so that the buoyant force per unit volume in the x^* direction is $g\Delta\rho/\Phi$. This is the crucial point of the analysis: the difficulty discussed by Cheung and Epstein has to do with the melting from below of a *horizontal* surface, for which Φ would approach infinity and the analysis given here would not apply. However, for any finite value of Φ, even if large—for example, for a nearly horizontal melting surface—$g\Delta\rho/\Phi$ is an appropriate *estimate* of the buoyant force per unit volume in the thermal boundary layer: since the whole argument is only a scaling one, a reasonable estimate is all that one needs to proceed. By using the already established relationships between scales, the x-momentum balance equation becomes, with ρ_W the water density:

$$v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial y} = Pr\frac{\partial^2 v}{\partial y^2} + \frac{Y^4 g\Delta\rho}{\rho_W \alpha_W^2 L_0 \Phi^2} \tag{25}$$

With the Prandtl number Pr for seawater being 7.1, the lefthand side, while not entirely negligible, is significantly less than the first term on the righthand side: within the thermal boundary layer, viscous forces are larger than inertia forces. Hence, the buoyant forces must be of the same order of magnitude as the viscous forces, say the second term on the righthand side of Eq. 25 is of the order of the Prandtl number. This establishes the scale Y. In particular, the ratio L_0/Y, which is recognized as being of the order of the Nusselt number based on L_0, is given by (k_W is the seawater heat conductivity and Gr the Grashof number based on $\Delta\rho$):

Table 1. Physical Properties for Seawater and Ice Used in the Calculations*

	Thermal Conductivity kcal/s·m·°C	Density kg/m^3	Specific Heat kcal/kg·°C	Thermal Diffusivity m^2/s
ICE	5.73×10^{-4}	915	0.50	1.25×10^{-6}
WATER	1.42×10^{-4}	1,023	1.01	1.37×10^{-6}

*Latent heat: 79.8 kcal/kg

$$Nu_L=\frac{hL_0}{k_W}\approx\left(\frac{GrPr}{\Phi^2}\right)^{1/4} \tag{26}$$

The values of the relevant physical properties are given in Table 1; these yield $Gr=2.30\times10^{20}$. (Note that, since the buoyant forces are dominated by differences in salinity, the Grashof number does not, as a first approximation, depend on T_W-T^*.) Since the Grashof number is large, buoyant forces are essentially balanced by inertia forces, and thus one would calculate a Reynolds number based on the characteristic scale L_0 as $Re_L=(Gr/\Phi)^{1/2}=1.81\times10^9$. This corresponds to a velocity of 1.77 m/s, well above the estimates for prevailing currents and for tidal pumping, thus indicating that free convection is presumably the dominant heat-transfer phenomenon.

However, the calculated value of the Reynolds number poses a problem. The analysis given above is based on the assumption that the flow within the boundary layer is laminar; this should be checked against the value of the Reynolds number based on the *momentum* boundary layer thickness δ, Re_δ. Y is the thickness of the *thermal* boundary layer, and therefore $\delta\approx YPr$. One therefore calculates:

$$Re_\delta=Pr^{3/4}Gr^{1/4}=5.35\times10^5 \tag{27}$$

This result implies that the boundary layer is in turbulent flow and therefore Eq. 26 cannot be used to estimate the heat-transfer coefficient. Assuming that the critical value of Re_δ for which the flow within the boundary layer becomes turbulent is 2,000, the corresponding critical value of the Grashof number, Gr_C, can be calculated as 4.47×10^{10} from the following equation:

$$2{,}000=Pr^{3/4}Gr_C^{1/4} \tag{28}$$

Equation 26 can therefore be used directly only up to a Grashof number value Gr_C. However, all available correlations for free convection indicate that, also when the boundary layer gets turbulent, the Nusselt number depends only on the product $GrPr$, and in particular it is proportional to its cubic root. (This is in agreement with the well known fact that, for heat transfer in turbulent flow past a wall, the exponent on the Prandtl number is 1/3.) Furthermore, existing correlations show that the Nusselt number does not undergo a discontinuity at the critical Grashof number. It follows that, if Nu_L is the Nusselt number calculated from Eq. 26, the Nusselt number at Gr values exceeding Gr_C is given by:

$$Nu=Nu_L(Gr/Gr_C)^{1/12} \tag{29}$$

Equations 26 and 29 yield, for the case at hand, a Nusselt number of 1.55×10^5, corresponding to a heat-transfer coefficient $h=2.20\times10^{-2}$ kcal/m^2·s·°C.

Length of the Freezing and the Melting Zones

The dimensional critical time can be expressed as follows:

$$t_C=\frac{\tau_C k^2}{\alpha h^2} \tag{30}$$

We now have estimates of all the relevant parameters, and these yield $t_C=6.57\times10^5$ s $=0.021$ yr. This corresponds to a length of 0.015 km, indicating that freezing occurs only on a very short region near the glacier end of the Drygalski Ice Tongue. Transient heat conduction in the ice rapidly quenches itself, and the dominant phenomenon becomes melting due to heat transfer to the water.

At the end of the freezing period, the thickness of the additional layer of ice resulting from the freezing phenomenon can be calculated from Eq. 6 to be about 0.12 m. It follows that, to all practical purposes, the initial freezing phenomenon can be neglected altogether, so that the problem becomes very easy. One simply has:

$$-\frac{ds}{dt}=\frac{\dot{H}\epsilon}{St} \tag{31}$$

We have now come to the point where we need an accurate estimate of the difference T_W-T^*. As far as T^* is concerned, there are two effects to consider (Foldvik and Kwinge, 1974). The effect of pressure is regulated by the Clausius-Clapeyron law. This yields a decrease of T^* with depth from sea level of 0.75°C/km. The other effect is due to salinity: at 34.5 wt. % salinity and atmospheric pressure, $T^*=-1.90$°C. We thus conclude that T^* is -1.90°C at sea level, and -2.65°C at the deepest point (the glacier side of the DIT).

Data on the average yearly temperature of surface seawater in the Terranova Bay give T_W between 0 and -2°C. The scatter is too large to allow us any meaningful estimate of T_W-T^*. The best data of water temperatures near Antarctic glaciers which we were able to obtain are those of Foldvik and Kvinge (1977), who report measured temperatures and salinities alongside the Filchner Ice Shelf at Lat. 77° 44′ S, Long. 41° 44′ W, up to a depth of 600 m. Salinity is 34.1% at the surface, but it rapidly grows to 34.5% at 400 m and stays constant at deeper levels; temperature is -1.8°C at the surface and stays approximately constant down to more than 300 m. A colder layer (-2.2°C) extends from about 400 m downward. This means that, at least at the conditions prevailing along the Filchner Ice Shelf, T_W-T^* is 0.1°C at the surface conditions and 0.45°C at the bottom. Given the uncertainties involved, an arithmetic mean of 0.28°C seems to be the best possible estimate.

To account for the difference between the volumetric flow rate of ice into the Drygalski Ice Tongue (14 km^3/yr) and the increase rate of the total volume of the DIT in the last 30 years (10 km^3/yr), we need a total volumetric rate of basal melting of 4 km^3/yr. Since the total exposed basal surface is approximately 1,400 km^2, this corresponds to a linear melting rate $ds/dt=-2.86$ m/yr. Equation 31 yields this value if one takes

$T_W - T^* = 0.30°C$, which is in remarkably good agreement with the estimated value of 0.28°C. One therefore may conclude that our simple model of heat transfer accounts for the correct order of magnitude of the average basal melting rate of the Drygalski Ice Tongue.

It is also of interest that the 4-km^3/yr basal melting rate constitute 28.6% of the ice flow rate into the Drygalski Ice Tongue. This compares extremely well with the following conclusion reported by Robin (1979): "It appears likely that something like 30% of the volume of the average Antarctic ice shelf is missing due to basal melting... In other words, if we apply our upward revision of the amount of basal melting to mass-balance calculations such as that of Bull (1971), we find the Antarctic ice sheet is close to being in balance between accumulations and losses."

Conclusions

A simple model for the heat-transfer phenomenon responsible for the average basal melting rate of the Drygalski Ice Tongue has been presented. Heat transfer appears to occur by free convection driven by salinity differences. The model contains no adjustable parameters, and it is surprisingly successful in predicting the *average* basal melting rate. Two observations of a general nature are, however, of importance.

First, the model presented shows how crucial it is to have very accurate measurements of the yearly average of the seawater temperature, T_W, as a function of depth. The driving force is only a fraction of a degree, and hence T_W needs to be known with an accuracy of 0.01°C.

The second point is that the model calculates only the average basal melting rate. In actual fact, the Foldvik and Kvinge (1977) data for the Filchner Ice Shelf seem to indicate that there is an intermediate region, about halfway along the ice shelf length, where T_W is in fact *less* than T^* (if only marginally), thus resulting locally in basal freezing. This does not invalidate the calculation of the average basal melting rate given above. Some evidence of the existence of an intermediate region where basal freezing occurs is in fact available (Thomas, 1979).

To conclude, it is worthwhile to note that free convection on lateral surfaces of the DIT was ignored in the model. This does not invalidate the results, since lateral melting amounts at most to 20% of the basal one.

Notation

- g = gravity acceleration, m^2/s
- Gr = Grashof number based on salinity-induced density difference, $(\rho g \Delta\rho L_0^3)/\mu^2$
- Gr_C = critical value of Gr at which flow in the boundary layer becomes turbulent
- h = water side heat-transfer coefficient, cal/m^2·s·°C
- H = h divided by ice density and specific heat, m/s
- k = thermal conductivity of ice, cal/m·s·°C
- k_W = thermal conductivity of water, cal/m·s·°C
- K = constant appearing in Eq. 6
- L = thickness of the Tongue, m
- L_0 = thickness at glacier side, m
- M = net rate of loss of ice, m/s
- Nu = Nusselt number, $= (hL_0)/k_W$
- Nu_L = Nu value for laminar boundary layer
- Pr = Prandtl number for seawater
- Re_L = Reynolds number for water flow, $(\rho_W V L_0)/\mu_W$
- Re_δ = Reynolds number for boundary layer, $(\rho_W V \delta)/\mu_W$
- $s(t)$ = position of ice-water interface, m
- St = Stefan number
- t = time, s
- t_C = critical time, s
- T_0 = initial temperature of ice, °C
- T^* = freezing temperature, °C
- T_W = water temperature, °C
- u = dimensionless distance into ice layer
- U = ice velocity in the tongue, m/yr
- v = dimensionless velocity along surface
- v^* = velocity along surface, m/s
- V = velocity scale along surface, m/s
- w = dimensionless velocity orthogonal to surface
- w^* = dimensional velocity orthogonal to surface, m/s
- W = velocity scale orthogonal to surface, m/s
- x = dimensionless coordinate along surface
- x^* = dimensional coordinate along surface, m
- X = global length of the Tongue, m
- y = dimensionless coordinate away from surface
- y^* = dimensional coordinate away from surface, m
- Y = thermal boundary layer thickness, m

Greek letters

- α = heat diffusivity in ice, m^2/s
- α_W = heat diffusivity in water, m^2/s
- β = dimensionless external temperature
- Γ = latent heat, kcal/kg
- δ = thickness of momentum boundary layer, m
- ϵ = ratio of external and internal temperature differences
- ϑ = dimensionless internal temperature
- μ = dimensionless position of interface
- μ_W = water viscosity, kg/m
- ρ = see ice density, kg/m^3
- ρ_W = water density, kg/m^3
- $\Delta\rho$ = density difference driving convection, kg/m^3
- τ = dimensionless time
- τ_C = dimensionless critical time
- Φ = ratio of length to depth
- Ω = frequency of temperature cycle, s^{-1}

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Appendix

Suppose the ice axial velocity U is regarded as being constant along the flow direction x^*, which is certainly a reasonable approximation. Let $L(x^*)$ be the local vertical thickness of the tongue, and M the net rate of loss of ice per unit exposed surface. The unsteady mass balance reads:

$$U\frac{\partial L}{\partial x^*}+\frac{\partial L}{\partial t}+M=0 \qquad \text{(A1)}$$

Equation A1 admits a solution with a triangular vertical cross section, $dL/dx^* = -1/\Phi$, under steady-state conditions, which would imply that $1/\Phi = M/U$ or, equivalently, that the total length of the tongue $X=\Phi L_0$ equals UL_0/M, with L_0 the glacier side thickness of the tongue. Now UL_0 is at least 0.7 km^2/yr, and M has been estimated at 0.00286 km/yr, so that one would get a minimum estimate for the steady-state length of the tongue of 245 km, which is almost four times the present length. One therefore concludes that the phenomenon which is being observed is certainly an unsteady-state one.

Equation A1 does not admit solutions with a triangular vertical cross section under unsteady-state conditions. Hence, one concludes that our analysis is based, as discussed earlier, on an approximation which uses the average value of Φ. The approximation, however, is a reasonable one, since the thickness data in Figure 2 support the hypothesis that the actual cross section is at least approximately triangular.

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